Complex Analysis: Final Exam

Aletta Jacobshal 01, Wednesday 27 January 2016, 18:30 – 21:30
Exam duration: 3 hours

Instructions — read carefully before starting

- Do not forget to very clearly write your **full name** and **student number** on each answer sheet and on the envelope. Do not seal the ennvelope.
- The exam consists of 6 questions; answer all of them.
- The total number of points is 100 and 10 points are "free".. The exam grade is the total number of points divided by 10.
- Solutions should be complete and clearly present your reasoning. If you use known results (lemmas, theorems, formulas, etc.) you must explain why the conditions for using such results are satisfied.
- You are allowed to have a 2-sided A4-sized paper with handwritten notes.

Question 1 (12 points)

(a) (6 points) Verify that the exponential function $f(z) = e^z$ satisfies the Cauchy-Riemann equations.

(b) (6 points) Show that the Taylor series of the exponential function $f(z) = e^z$ around $z_0 \in \mathbb{C}$ is given by

$$e^z = e^{z_0} \sum_{j=0}^{\infty} \frac{(z - z_0)^j}{j!}.$$

What is the domain where the Taylor series of e^z around z_0 converges? [If necessary, you can use without proof the Taylor series of e^z around 0.]

Question 2 (18 points)

Consider the function

$$f(z) = \frac{e^{iz}}{z^2 - 4}.$$

(a) (6 points) Compute the residue of $f(z)$ at each one of the singularities of $f(z)$.

(b) (12 points) Compute the principal value

$$\text{pv} \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 - 4}\, dx.$$

Question 3 (14 points)

Consider the branch $f(z) = e^{\frac{1}{3}\mathcal{L}_0(z)}$ of the cubic root function $z^{\frac{1}{3}}$. The branch $\mathcal{L}_0(z)$ of the logarithm has a branch cut along the positive real axis.

(a) (6 points) Compute $f(-i)$ and $f'(-i)$. Write the results in Cartesian form.

(b) (8 points) Evaluate the limits $\lim_{\varepsilon \to 0^+} f(x + i\varepsilon)$ and $\lim_{\varepsilon \to 0^+} f(x - i\varepsilon)$ for $x > 0$. Express the limits in terms of $\sqrt[3]{x}$, that is, the positive real cubic root of the real number $x > 0$.

Question 4 (14 points)

Consider the function

$$f(z) = \frac{1}{z-1} + \frac{2}{2-z}.$$

(a) (4 points) Determine the singularities of $f(z)$ and their type.

(b) (10 points) Compute the Laurent series at 0 of the function in the domain $1 < |z| < 2$.

Question 5 (16 points)

(a) (6 points) Given the function

$$f(z) = \frac{(z-4)(z-1)^2 \sin z}{z^2+1},$$

evaluate the integral

$$\int_C \frac{f'(z)}{f(z)}\, dz,$$

where C is the positively oriented circular contour with $|z| = 2$.

(b) (10 points) Use Rouché's theorem to show that the polynomial $P(z) = \varepsilon z^3 + z^2 + 1$, where $0 < \varepsilon < 3/8$, has exactly 2 roots in the disk $|z| < 2$.

Question 6 (16 points)

(a) (8 points) Show that

$$\left| \int_C \frac{e^z}{z+1}\, dz \right| \leq 2\pi e^2,$$

where C is the positively oriented circle $|z-1| = 1$.

(b) (8 points) Suppose that $f(z)$ is an analytic function in a domain D and it has no zeros in D. Show that if $|f(z)|$ attains its minimum in D (that is, there exists a point $z_0 \in D$ such that $|f(z_0)| \leq |f(z)|$ for all $z \in D$), then $f(z)$ is constant.

End of the exam (Total: 90 points)